

Gauge theories with non-unitary parallel transporters: a soluble Higgs model*

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Abstract. It has been proposed to abandon the requirement that parallel transporters in gauge theories are unitary (or pseudo-orthogonal). This leads to a geometric interpretation of Vierbein fields as parts of gauge fields, and non-unitary parallel transport in extra directions yields Higgs fields. In such theories, the holonomy group H is larger than the gauge group G . Here we study a one-dimensional model with fermions which retains only the extra dimension, and which is soluble in the sense that its renormalization group flow may be exactly computed, with $G = SU(2)$ and non-compact $H \subseteq GL(2, \mathbf{C})$, or $G = U(2)$, $H = GL(2, \mathbf{C})$. In all cases the asymptotic behavior of the Higgs potential is computed, and with one possible exception for $G = SU(2)$, $H = GL(2, \mathbf{C})$, there is a flow of the action from a UV fixed point which describes a $SU(2)$ -gauge theory with unitary parallel transporters, to a IR fixed point. We explain how exponential mass ratios of fermions of different flavor can arise through spontaneous symmetry breaking, within the general framework.

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1 Introduction

This paper is part of a larger study to explore different possibilities of giving a geometric interpretation to Higgs fields by unifying with gauge fields. This theme is of interest to model builders [1]. They study gauge theories in space-time of five or more dimensions, often with boundaries called branes. The associated four-dimensional effective theory is supposed to exhibit a Higgs potential. We are interested in how such Higgs potentials can arise as a result of a sequence of renormalization group (RG) transformations.

By definition, effective field theories have a UV-cutoff M with the dimension of a mass, and M can be lowered by a renormalization group transformation [2]. Let us assume that the cutoff is introduced in the form of a lattice. When the lattice spacing comes down to the extension of space-time in the extra direction(s), an effective four-dimensional theory results.

Our hypothesis is that Higgs fields appear in higher dimensional field theories in the following way. At the level of effective theories, they are (parts of) non-unitary parallel transporters ϕ along links of the lattice in the extra direction or directions. The parallel transporters take their values in a group H , typically non-compact, which is larger than the gauge group G . They can arise from unitary parallel transporters, i.e. ordinary gauge theories without Higgs fields, through real space renormalization group transfor-

mations. Related ideas have been developed in the context of deconstruction [3]. We are particularly interested in theories on a five-dimensional space-time with two boundaries, where left handed fermions may live on one of them and right handed fermions on the other.

In gauge theories, parallel transporters $\mathcal{U}(C)$ along paths C are required by the principles and are determined by vector potentials. The lattice gauge fields of lattice gauge theory are parallel transporters. Traditionally they are unitary maps between vector spaces with scalar products. It has been proposed [4] to abandon the additional requirement of their unitarity (or pseudo-orthogonality), $\mathcal{U}(C)^* = \mathcal{U}(C)^{-1}$. Admitting non-unitary parallel transporters appears natural in the light of the general theory [5, 6] of complex systems, and in view of the fact that the requirement of their unitarity finds no natural place in discrete differential calculus and geometry [7].

This generalization leads to a geometric interpretation of Vierbein fields as parts of gauge fields, in four dimensions, while non-unitary parallel transport in extra directions yields Higgs fields. One gets, besides the unitary gauge group G , a larger holonomy group $H \supset G$ with an involutive automorphism θ such that $\theta(g) = g$ if and only if $g \in G$. Only G is a local symmetry, but fields form representation spaces of H . In lattice models, the lattice gauge fields take their values in H . Here we study a one-dimensional model with fermions which retains only the extra dimension, and which is soluble in the sense that its renormalization group flow may be exactly computed, with $G = SU(2)$, and

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$H \subseteq GL(2, \mathbf{C})$, or $G = U(2)$, $H = GL(2, \mathbf{C})$. A summary of results is found in Sect. 4.

The parametrization¹ of parallel transporters in extra directions as $\phi = e^{\Xi}U$, $U \in G$ exhibits Higgs fields Ξ which take their values in the tangent space \mathfrak{p} to the coset space H/G . This coset space is a symmetric space in case G is the maximal compact subgroup of H , and it is determined by an involutive automorphism of the compact real form H_c of the complexification of H [8]. \mathfrak{p} is isomorphic to the non-compact part of the Lie algebra of H in this case.

There is a remarkable mathematical correspondence with another geometrization of the Higgs fields, where one identifies the Higgs fields with conventional gauge fields in the extra direction(s), and some difficulties associated with this possibility are overcome by orbifolding. According to the group theoretical classification of Hebecker and Ratz [9], the possibilities are again classified by involutive automorphisms of the original gauge group H_c , in the case of division by \mathbf{Z}_2 factors. Here, H_c is broken to G .

The physical correspondence is not understood. Finite dimensional representations and invariants are the same for H and H_c , by Weyl's unitary trick. But a RG-study of Higgs potentials is only possible in our scheme which is based on non-unitary parallel transporters. Moreover, this scheme can lead to exponential mass ratios. This is explained in Sect. 3. A five-dimensional model with mass splitting between generations is proposed in [10].

To illuminate the essence of the RG-mechanism, consider how qualitative features of models can be understood in terms of properties of RG-flows in other cases.

Consider mass generation in the $O(N)$ -symmetric two-dimensional non-linear σ -model. The fields $\mathbf{s}(z)$ in the fundamental Lagrangean take their values on the sphere S^{N-1} . A real space RG requires a choice of block spins $\phi(x)$. They are the fields of the effective theory on a block lattice whose sites x are squares of the original space with points z . One may choose $\phi(x)$ as an average of $\mathbf{s}(z)$ over the square x . But then, $\phi(x)$ will no longer take its value in the sphere, but has fluctuating length $|\phi(x)|$, and a $O(N)$ -invariant potential $V(\phi(x))$ will appear in the effective action. In principle one could integrate out the length, returning to fields on the sphere. But this ceases to be permitted when the potential becomes flat at $\phi = 0$ at some length scale, because the integration would then lead to non-localities in the effective action. The appearance of a mass becomes manifest when at some still larger length scale the potential V becomes single well.

Confinement in four-dimensional gauge theories arises in an analogous fashion when the effective theory becomes a dielectric gauge theory [11, 12]. The effective lattice action [12] depends on non-unitary parallel transporters ϕ , and electric flux tubes with energy proportional to their length form when the potential $V(\phi)$ has its absolute minimum at $\phi = 0$ (" $\Xi = -\infty$ "). At finite temperature T , i.e. finite extension of the lattice in the fourth direction, the

potentials V may become T dependent, and deconfinement may result.

In the case of interest in this paper, the Higgs potential V may be regarded as a function $\mathcal{V}(\Xi)$ on \mathfrak{p} which is in fact determined by its values on a maximal abelian Lie algebra $\mathfrak{a} \subset \mathfrak{p}$. When one starts from an ordinary gauge theory, \mathcal{V} starts with extreme single well shape and one hopes for RG-flow to a shape with multiple finite minima on \mathfrak{a} . If so, flavor mass splitting results (Sect. 3). The one-dimensional model exhibits a non-trivial RG-flow, but it turns out to be too simple to demonstrate the appearance of multiple minima.

2 The one-dimensional model

We consider a one-dimensional Euclidean lattice model with fermions. It lives on a chain of lattice spacing $a_0 = 1$ whose sites are labeled by $z \in \mathbf{Z}$.

The Fermi fields $\psi(z)$, $\bar{\psi}(z)$ sit on the sites of the chain Λ^0 . They have only one spin component, but two color-components. Elements ϕ of $H \subseteq GL(2, \mathbf{C})$ can act on them. They transform according to the fundamental representation of the gauge group $G = SU(2)$. As usual, ψ and $\bar{\psi}$ are treated as independent integration variables in the Feynman-Kac path integral [16].

With every link $(z, z+1)$ and its adjoint $(z+1, z)$ between neighboring sites there is associated a possibly non-unitary lattice gauge field (parallel transporter) $\phi(z+1, z) \in H$ and its adjoint,

$$\phi(z, z+1) = \phi(z+1, z)^*.$$

The groups of interest are $G = SU(2)$,

$$H = \mathbf{R}_+SU(2), SL(2, \mathbf{C}), \mathbf{R}_+SL(2, \mathbf{C}), GL(2, \mathbf{C}).$$

In Sect. 12 the results will be extended to

$$G = U(2), H = GL(2, \mathbf{C}).$$

Under a $SU(2)$ gauge transformation,

$$\begin{aligned} \psi(z) &\mapsto u(z)\psi(z), \\ \phi(z+1) &\mapsto u(z+1)\phi(z+1, z)u(z)^{-1}. \end{aligned}$$

The initial action is

$$\begin{aligned} S^0(\psi, \bar{\psi}, \phi) &= \sum_z \left(S_F^0(z, z+1) - V(\phi(z, z+1)) \right), \\ S_F^0(z, z+1) &= \mathfrak{z} \bar{\psi}(z)\phi(z, z+1)\psi(z+1) \\ &\quad + \bar{\mathfrak{z}} \bar{\psi}(z+1)\phi(z+1, z)\psi(z). \end{aligned} \quad (1)$$

The constant factor \mathfrak{z} and its complex conjugate $\bar{\mathfrak{z}}$ are included for later convenience. $V(\Phi)$ is interpreted as a Higgs potential. It is a real function of $\Phi \in H$ with the following further property, which will be shared by not necessarily real coefficients in the effective actions:

$$V(\Phi^*) = \overline{V(\Phi)}. \quad (2)$$

¹ This parametrization is suitable for taking the continuum limit. If there is only one extra direction, the factor U can be gauged away.

The action is required to be invariant under $SU(2)$ -gauge transformations. Therefore the Higgs potential has the *bi-invariance property*

$$V(u_1\Phi u_2) = V(\Phi) \text{ for all } u_1, u_2 \in G, \Phi \in H. \quad (3)$$

The path integral involves the Haar measure $d\Phi$ on H to integrate over the lattice gauge fields $\phi(z+1, z)$. We assume that $e^{-V(\Phi)}$ is integrable and square integrable on the group H ; hence

$$V(\Phi) \mapsto +\infty \text{ when } \|\Phi\| \mapsto \infty.$$

The initial Boltzmann factor is

$$\begin{aligned} B^0(\psi, \bar{\psi}, \phi) &= e^{S^0} = \prod_z B^0(z, z+1) \\ &= \prod_z B_F^0(z, z+1) e^{-V^0(\phi(z, z+1))}, \end{aligned}$$

where $B^0(z, z+1)$ depends on $\psi(z), \bar{\psi}(z), \psi(z+1), \bar{\psi}(z+1), \phi(z, z+1)$.

For the groups studied here, $\Phi \in H$ can be parametrized as $\Phi = u_1 d(\eta) u_2$, $u_i \in G$, and $d(\eta)$ a diagonal matrix depending on a parameter η in a linear space \mathfrak{a} . Hence

$$V(\Phi) = \mathcal{V}(\eta) \quad (4)$$

by G -bi-invariance. There is always a discrete group W_T of linear transformations $\eta \mapsto w(\eta)$ of \mathfrak{a} which are induced by transformations in G , viz. $d(w(\eta)) = g d(\eta) g^{-1}$ for suitable $g \in G$. It follows that

$$\mathcal{V}(\eta) = \mathcal{V}(w(\eta)), \quad w \in W_T. \quad (5)$$

We will show in Sect. 3 that flavor mass splitting results when this symmetry under W_T is broken spontaneously, i.e. if the orbits of minima of \mathcal{V} do not consist of a single point.

In the rest of this paper we study the real space renormalization group (RG) flow of this one-dimensional model. After n renormalization group steps, the model lives on a chain $\Lambda^n \subset \Lambda^0$ of lattice spacing $a_n = L^n a_0$ and depends on fields $\psi^n, \bar{\psi}^n$ on the sites of the block lattice Λ^n and on lattice gauge fields $\phi^n(z, z+a_n)$ sitting on the links. Somewhat surprisingly, it turns out that the block side L has to be chosen odd, and we choose $L = 3$. For $L = 2$, the effective Boltzmann factor would not admit taking the logarithm to obtain the effective action; see Sect. 6.2.

The block spin transformation will act on the fermions by decimation, so that

$$\psi^n(z) = \psi(z) \text{ for } z \in \Lambda^n \quad (6)$$

and similarly for $\bar{\psi}$. For the gauge fields, the appropriate choice of block spin is

$$\begin{aligned} \phi^{n+1}(z+3a_n, z) &= \phi^n(z+3a_n, z+2a_n)\epsilon \\ &\quad \times \phi^n(z+a_n, z+2a_n)^T \epsilon^T \\ &\quad \times \phi^n(z+a_n, z), \end{aligned}$$

for $n = 0, 1, 2, \dots$, with $\phi^0 = \phi$, $a_0 = 1$. The expression involves the antisymmetric tensor ϵ in two dimensions, with $\epsilon_{12} = 1$, and A^T stands for the transpose of the matrix A . In particular

$$\begin{aligned} \phi^1(z+3, z) &= \phi(z+3, z+2)\epsilon \\ &\quad \times \phi(z+1, z+2)^T \epsilon^T \phi(z+1, z), \end{aligned} \quad (7)$$

Under the RG-transformation steps considered here, the effective actions will remain strictly local in the sense that they are sums of contributions from block links,

$$S^n(\psi^n, \bar{\psi}^n, \phi^n) = \sum_{z \in \Lambda^n} S^n(z, z+a_n),$$

with $S^n(z, z+a_n)$ depending only on $\psi^n(z), \bar{\psi}^n(z), \psi^n(z+a_n), \bar{\psi}^n(z+a_n)$, and $\phi^n(z, z+a_n)$.

When H is not a subgroup of $SL(2, \mathbf{C})$, the effective actions are more complicated than (1). Integration of the Higgs fields ϕ produces 4-fermion, 6-fermion and 8-fermion interactions, with ϕ -dependent coefficients. We present recursion relations for the corresponding coefficients, and their solution. The coefficients in the effective action are obtained by taking the logarithm. In case $H = SL(2, \mathbf{C})$ there is a great simplification, so that this model is nearly trivial. The effective action retains the form (1), and the flow of the effective Higgs potential V receives no contributions from the fermions.

3 Generation of flavor mass splitting

We wish to explain how non-unitary parallel transporters may lead to exponential mass splitting.

For definiteness consider first $H = SL(2, \mathbf{C})$, $G = SU(2)$. Matrices $A \in SL(2, \mathbf{C})$ may be parametrized as $A = u_1 d(\eta) u_2$ with $u_i \in G$, η real and $d(\eta) = \text{diag}(e^{-\eta/2}, e^{\eta/2})$. There is a $w \in G$ such that $wd(\eta)w^{-1} = d(-\eta)$; cf. Lemma 3 of Sect. 7. The bi-invariance property (3) implies that $\mathcal{V}(\eta) = V(d(\eta))$ obeys

$$\mathcal{V}(\eta) = \mathcal{V}(-\eta).$$

Suppose that the orbit of the minima under the discrete group $W_T = \{1, -1\}$ is non-trivial; then the minima of $\mathcal{V}(\eta)$ are at $\pm\hat{\eta} \neq 0$. Then at the minima of $V(\phi)$, $\phi = u_1 d(\hat{\eta}) u_2$ and the interaction term becomes

$$3\bar{\Psi}\phi\Psi = \sum_{\alpha=1}^2 \bar{\Psi}'_{\alpha} m_{\alpha} \Psi'_{\alpha}, \quad (8)$$

with $\bar{\Psi}' = \bar{\Psi} u_1$, $\Psi' = u_2 \Psi$, and masses

$$m_1 = 3e^{-\hat{\eta}/2}, \quad m_2 = 3e^{\hat{\eta}/2}.$$

We see that we have fermions of two flavors with different masses.

Although the one-dimensional model does not show this feature, we hope that in realistic models there is a RG-flow

from an infinitely deep single well \mathcal{V} as corresponds to ordinary gauge theory to \mathcal{V} with multiple minima η . If the RG-flow is stopped there because the extra dimension has finite extent, we get flavor mass splitting.

Let us explain how this generalizes to other groups G , assuming G is the maximal compact subgroup of H . In the notation of this paper, which is non-standard in group theory, the Cartan decomposition of the Lie algebra \mathfrak{h} of H reads $\mathfrak{h} = \mathfrak{g} + \mathfrak{p}$, where \mathfrak{g} is the Lie algebra of G , and $[\mathfrak{h}, \mathfrak{p}] \subseteq \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{g}$. At the group level, this translates into the decomposition $\phi = e^{\Xi} U$ of parallel transporters $\phi \in H$ in the extra direction(s), with $U \in G$, $\Xi \in \mathfrak{p}$. When non-zero, Ξ may be interpreted as a Higgs field. The Higgs potential $V(\phi)$ may be regarded as a function $\mathcal{V}(\Xi)$ on \mathfrak{p} which is invariant under $\Xi \mapsto g\Xi g^{-1}$ for $g \in G$.

\mathfrak{p} contains the non-compact part \mathfrak{a} of a maximally non-compact Cartan subalgebra of \mathfrak{h} . Because every element of \mathfrak{p} is conjugate under G to an element η of \mathfrak{a} , the Higgs potential may equally be regarded as a function $\mathcal{V}(\eta)$ on \mathfrak{a} which is invariant under elements of the so-called restricted Weyl group W_T . W_T consists of transformations $\mathfrak{a} \mapsto \mathfrak{a}$ which are of the form² $\eta \mapsto w(\eta) = \text{Ad}(g)\eta$ for suitable $g \in G$.

Spontaneous symmetry breaking and mass splitting occurs when non-trivial orbits of minima $\hat{\eta}$ of $\mathcal{V}(\eta \in \mathfrak{a})$ under W_T appear. Since \mathfrak{a} is commutative, the minima $\hat{\eta}$ may be regarded as diagonal matrices, with entries $\hat{\eta}_i$. The masses depend exponentially on them,

$$m_i = \mathfrak{z} e^{\hat{\eta}_i}.$$

A classification of possible unbroken subgroups of G in terms of properties of the restricted roots (or parabolic subgroups) of H was derived by de Riese [13] for simple groups H . Symmetry breaking to a discrete subgroup is possible when \mathfrak{a} is a completely non-compact Cartan subalgebra of \mathfrak{h} [14].

4 Summary of results on the RG-flow

4.1 General form of the effective action

In case $H \neq SL(2, \mathbf{C})$, non-bilinear fermion actions will be produced by the RG-flow. It follows from the Fierz identities, Lemma 1, that the most general action has the form

$$\begin{aligned} S^n(z, z + a_n) &= c_0^n + c_1^n L^n + \bar{c}_1^n \bar{L}^n + c_2^n k^n + \bar{c}_2^n \bar{k}^n + c_3^n L^n \bar{L}^n \\ &\quad + c_4^n k^n \bar{L}^n + \bar{c}_4^n \bar{k}^n L^n + c_5^n k^n \bar{k}^n, \end{aligned} \quad (9)$$

where

$$\begin{aligned} L^n &= \bar{\psi}(z) \phi^n(z, z + a_n) \psi(z + a_n), \\ \bar{L}^n &= \bar{\psi}(z + a_n) \phi^n(z + a_n, z) \psi(z), \end{aligned} \quad (10)$$

² Ad is the adjoint action of H on its Lie algebra \mathfrak{h} ; in terms of matrices $\text{Ad}(g)\eta = g\eta g^{-1}$.

$$k^n = \frac{1}{4} \bar{\psi} \epsilon \bar{\psi}(z) \times \psi \epsilon \psi(z + a_n), \quad (11)$$

$$\bar{k}^n = \frac{1}{4} \bar{\psi} \epsilon \bar{\psi}(z + a_n) \times \psi \epsilon \psi(z),$$

and where c_i^n are functions of $\phi^n(z, z + a_n) = \phi^n(z + a_n, z)^*$; \bar{c}_i^n is the complex conjugate of c_i^n , and c_0^n, c_3^n and c_5^n are real. In particular, c_0^n is the negative effective Higgs potential

$$c_0^n = -V^n(\phi^n(z, z + a_n)).$$

We identify $V^0 = V$.

The Boltzmann factor will have the same general form,

$$\begin{aligned} B^n(z, z + a_n) &= a_0^n + a_1^n L^n + \bar{a}_1^n \bar{L}^n + a_2^n k^n + \bar{a}_2^n \bar{k}^n + a_3^n L^n \bar{L}^n \\ &\quad + a_4^n k^n \bar{L}^n + \bar{a}_4^n \bar{k}^n L^n + a_5^n k^n \bar{k}^n, \end{aligned} \quad (12)$$

with $a_0^n = \exp(-V^n(\phi^n(z, z + a_n)))$.

In more careful notation we write

$$\begin{aligned} a_i^n &= a_i^n(z, z + a_n) = a_i^n[\phi^n(z, z + a_n)], \\ \bar{a}_i^n &= \bar{a}_i^n(z, z + a_n) = \overline{a_i^n[\phi^n(z, z + a_n)]}. \end{aligned}$$

The following reality requirements will be satisfied:

$$a_j^n[\Phi^*] = \bar{a}_j^n[\Phi]$$

and similarly for the coefficients c_j^n of the action. It follows from (2) that they are fulfilled for the initial values a_i^0 , and they will pass through the recursion relations.

4.2 Flow of the effective action for general H

Theorem 1 (RG-flow for general H) *The coefficients $a_i^n[\Phi]$ in the effective Boltzmann-factor obey the following recursion relations:*

$$\begin{aligned} a_0^{n+1}[\Phi] &= a_0^n * \bar{a}_5^n * a_0^n[\Phi], \\ a_1^{n+1}[\Phi] &= a_1^n * \bar{a}_4^n * a_1^n[\Phi], \\ a_2^{n+1}[\Phi] &= a_2^n * \bar{a}_2^n * a_2^n[\Phi], \\ a_3^{n+1}[\Phi] &= a_3^n * \bar{a}_3^n * a_3^n[\Phi], \\ a_4^{n+1}[\Phi] &= a_4^n * \bar{a}_1^n * a_4^n[\Phi], \\ a_5^{n+1}[\Phi] &= a_5^n * \bar{a}_0^n * a_5^n[\Phi], \end{aligned}$$

where $*$ is the convolution product on the group H , and $\Phi \in H$ ($\bar{a}_i^n = a_i^n$ for $i = 0, 3, 5$).

All the coefficients are $SU(2)$ -bi-invariant functions on H , viz. $a_i[u_1 \Phi u_2] = a_i[\Phi]$ for $\Phi \in H$, $u_1, u_2 \in SU(2)$.

The coefficients c_i^n in the effective action are related to a_i^n by Proposition 2 of Sect. 10.

We may turn the convolution product into a product by harmonic analysis on the group H .

For our groups H , every unitary irreducible representation contains at most one normalized $SU(2)$ -invariant

vector in its representation space. Let us denote by Ξ those representations which possess such a vector, and which contribute to the harmonic expansion of good functions (= square integrable on the group), and let us denote the restriction of the Plancherel measure to such representations by $d\Xi$.

Let us denote the corresponding matrix element of the representation operator by $P^\Xi(\Phi)$ and call it a spherical function. Good $SU(2)$ -bi-invariant functions on H admit an expansion in spherical functions, $f[\Phi] = \int d\Xi \tilde{f}(\Xi) P^\Xi(\Phi)$. Under harmonic analysis, convolutions turn into products. Therefore we have the following corollary of the recursion relations.

Corollary 1 *Write $\bar{a}_i = a_{\bar{i}}$. Then,*

$$a_i^{n+1}[\Phi] = \int d\Xi \tilde{a}_i^{n+1}(\Xi) P^\Xi(\Phi),$$

with Fourier coefficients

$$\tilde{a}_i^{n+1}(\Xi) = \sum_{jkl} d^i_{jkl} \tilde{a}_j^n(\Xi) \tilde{a}_k^n(\Xi) \tilde{a}_l^n(\Xi),$$

where

$$\tilde{a}_i^n(\Xi) = \int_H d\Phi a_i^n[\Phi] P^\Xi(\Phi^{-1}).$$

The sum over $j, k, l = 0, 1, \bar{1}, 2, \bar{2}, 3, 4, \bar{4}, 5$ has only one non-vanishing term, which is equal to 1. The non-vanishing coefficients are

$$d^0_{050}, d^1_{1\bar{4}1}, d^2_{2\bar{2}2}, d^3_{333}, d^4_{4\bar{4}4}, d^5_{505};$$

$$d^{\bar{1}}_{\bar{1}\bar{4}\bar{1}}, d^{\bar{2}}_{\bar{2}\bar{2}\bar{2}}, d^{\bar{4}}_{\bar{4}\bar{4}\bar{4}}.$$

The explicit description of Ξ and P^Ξ for the various groups H is found in Appendix A.

The result is also valid for $H = SL(2, \mathbf{C})$. It permits one to compute the renormalization group flow when one starts with a more general action than (1).

4.3 Flow between fixed points

The asymptotic behavior of the Higgs potential after many RG-steps can be computed. It becomes flat in the case $\mathbf{R}_+SU(2)$, and tends to a definite function on H in the other cases. This is also true for the model with $G = U(2), H = GL(2, \mathbf{C})$.

There is a UV fixed point which describes a $SU(2)$ -gauge theory with fermions with unitary parallel transporters $\phi(z, z + 1) \in SU(2)$,

$$e^{-V(\Phi)} = \delta_{SU(2)}(\Phi). \tag{13}$$

$\delta_{SU(2)}(\Phi)$ is a δ -function concentrated on $SU(2) \subset H$, so that $\int_H d\Phi f(\Phi) \delta_{SU(2)}(\Phi) = \int_{SU(2)} du f(u)$. Since the unitary parallel transporters may be transformed away in one dimension, this model is equivalent to a free fermion theory. The asymptotic behavior of the model after a large number n of RG-steps may be computed, starting from $e^{-V(\zeta d(\eta))}$

which may be arbitrarily close to the UV fixed point (13) and satisfies the following assumptions. It is square integrable on H , piecewise continuous, and $\int d\Phi |\zeta|^\epsilon e^{\epsilon|\eta|} e^{-V} < \infty$ for some $\epsilon > 0$.

In all cases, including $G = U(2), H = GL(2, \mathbf{C})$ but with the possible exception of $G = SU(2), H = GL(2, \mathbf{C})$, there is a flow away from this fixed point and towards a IR fixed point. In the exceptional case we are unable to compute the asymptotic behavior of some coefficients in the fermionic action other than the Higgs potential.

The case $G = SU(2), H = GL(2, \mathbf{C})$ is different from all the others in that G is not the maximal compact subgroup of H .

$\mathbf{H} = \mathbf{SL}(2, \mathbf{C})$

We use the parametrization $A = u_1 d(\eta) u_2$ of elements of $A \in SL(2, \mathbf{C})$, that will be given explicitly in Sect. 6.4.

Choose the additive constant in V such that

$$\int_{SL(2, \mathbf{C})} e^{-V(A)} dA = 1.$$

Let

$$b = \int \frac{1}{12} \sinh^2 \eta d\eta \eta^2 e^{-V(d(\eta))}$$

and $B = Nb, N = 3^n$. Then

$$e^{-V^n(d(\eta))} \sim \frac{1}{(2\pi B)^{3/2}} \frac{\eta}{4 \sinh \eta} \exp\left(-\frac{1}{8B} \eta^2\right). \tag{14}$$

The Gaussian factor tends to 1 as $n \mapsto \infty$ because $B \propto 3^n$.

Preliminaries for general H

We note that $a_0^0(\Phi)$ and $a_5^0(\Phi)$ are positive. By our assumptions, $a_0^0 d\Phi$ and $a_5^0(\Phi) d\Phi$ are therefore normalizable measures, and so is their convolution. Let the additive constant in V be chosen so that the measure $a_0^0 * a_5^0 d\Phi$ is normalized to one, and let $\mathfrak{N} = \int d\Phi e^{-V(\Phi)}$.

$\mathbf{R}_+ \mathbf{SU}(2)$

Elements of $\mathbf{R}_+SU(2)$ have the form ru with $r > 0$ and $u \in SU(2)$, but $e^{-V(ru)}$ and $\det(ru)$ are both independent of u . Let

$$a = \int \frac{dr}{r} \ln r (a_0^0 * a_5^0)(ru),$$

$$a' = \int \frac{dr}{r} \ln r a_0^0(ru),$$

$$b = \int \frac{dr}{r} \ln^2 r (a_0^0 * a_5^0)(ru),$$

$$b' = \int \frac{dr}{r} \ln^2 r a_0^0(ru),$$

$$\begin{aligned} A &= a' + Na, \\ B &= b' + Nb, \\ N &= \frac{1}{2}(3^n - 1). \end{aligned}$$

Then the asymptotic behavior of the Higgs potential is as follows:

$$e^{-V^n(ru)} \sim \frac{\mathfrak{N}}{(2\pi B)^{1/2}} \exp\left(-\frac{1}{2B} \ln^2(re^A)\right).$$

The constant A grows like N , but the term proportional N can be transformed away by a wave function renormalization. This is equivalent to modifying the block spin definition to

$$\begin{aligned} \phi^{n+1}(z + 3a_n, z) &= Z^{1/2} \phi^n(z + 3a_n, z + 2a_n) \epsilon \\ &\times \phi^n(z + a_n, z + 2a_n)^T \epsilon^T \phi^n(z + a_n, z), \end{aligned}$$

with a suitable n -independent constant Z . When A stays bounded, the exponential factor tends to 1 for large n , so that the Higgs potential becomes flat.

GL(2, C) and R₊SL(2, C)

The asymptotic behavior of $e^{-V(\zeta d(\eta))}$ is independent of the phase of ζ and combines the factors encountered in the previously considered cases:

$$\begin{aligned} e^{-V^n(\zeta d(\eta))} &\sim \frac{\mathfrak{N}}{(2\pi B)^{3/2} (2\pi B')^{1/2}} \frac{\eta}{4 \sinh \eta} \\ &\times \exp\left(-\frac{1}{8B} \eta^2 - \frac{1}{2B'} \ln^2(|\zeta|e^A)\right), \end{aligned}$$

with constants A, B' which are of order N with $N = \frac{1}{2}(3^n - 1)$ and B of order 3^n and can be computed from the starting values of V similarly as in the other cases. By a wave function renormalization, A can be transformed to $O(1)$, and the exponential then tends to a constant in the limit.

5 Initial Boltzmannian

To obtain starting values a_i^0 for the RG-flow for general H , we need to compute $B_F^0(z, z + 1) = e^{S_F^0(z, z+1)}$ for the action (1). We retain the definitions (10) of $L = L^0, \bar{L} = \bar{L}^0, k = k^0$ and $\bar{k} = \bar{k}^0$ and introduce

$$\begin{aligned} l &= l(z, z + 1) \\ &= \det(\phi(z, z + 1)) \bar{\psi} \epsilon \bar{\psi}(z) \times \psi \epsilon \psi(z + 1) \\ &= 4 \det(\phi(z, z + 1)) k(z, z + 1), \\ \bar{l} &= \bar{l}(z, z + 1) \\ &= \det(\phi(z + 1, z)) \bar{\psi} \epsilon \bar{\psi}(z + 1) \times \psi \epsilon \psi(z) \\ &= 4 \det(\phi(z + 1, z)) \bar{k}(z, z + 1). \end{aligned}$$

We will make use of the following identities.

Lemma 1 For $\Phi \in GL(2, \mathbf{C})$ and 2-component Grassmann variables $\psi \neq \bar{\psi}$,

$$(\bar{\psi} \Phi \psi)^2 = -\frac{1}{2} [\det \Phi] \bar{\psi} \epsilon \bar{\psi} \times \psi \epsilon \psi.$$

Proof. The lemma is an immediate consequence of the identities

$$\begin{aligned} \epsilon \phi^T \epsilon^{-1} &= [\det \phi] \phi^{-1}, \\ \psi_\alpha \psi_\beta &= \frac{1}{2} \epsilon_{\alpha\beta} \psi \epsilon \psi, \quad \bar{\psi}_\alpha \bar{\psi}_\beta = -\frac{1}{2} \epsilon_{\alpha\beta} \bar{\psi} \epsilon \bar{\psi}. \end{aligned} \tag{15}$$

It follows that

$$L^2 = -\frac{1}{2} l \quad \text{and} \quad \bar{L}^2 = -\frac{1}{2} \bar{l}.$$

The expansion of the exponential stops because products of more than two ψ 's or more than two $\bar{\psi}$'s at any site vanish. As a result one obtains

$$\begin{aligned} B_F^0(z, z + 1) &= 1 + \mathfrak{z}L + \bar{\mathfrak{z}}\bar{L} - \frac{1}{4} \mathfrak{z}^2 l + \mathfrak{z}\bar{\mathfrak{z}}L\bar{L} - \frac{1}{4} \bar{\mathfrak{z}}^2 \bar{l} \\ &\quad - \frac{1}{4} \mathfrak{z}^2 \bar{\mathfrak{z}}l\bar{L} - \frac{1}{4} \mathfrak{z}\bar{\mathfrak{z}}^2 \bar{l}L + \frac{1}{16} \mathfrak{z}^2 \bar{\mathfrak{z}}^2 l\bar{l}. \end{aligned} \tag{16}$$

For the coefficients in $B^0(z, z+1) = B_F^0(z, z+1) e^{-V(\phi(z, z+1))}$, this yields the following initial values:

$$a_i^0 = b_i^0 e^{-V^0} \tag{17}$$

and

$$\begin{aligned} b_0^0 &= 1, \\ b_1^0 &= \mathfrak{z}, \\ b_2^0 &= -\mathfrak{z}^2 \det \phi, \\ b_3^0 &= \mathfrak{z}\bar{\mathfrak{z}}, \\ b_4^0 &= -\mathfrak{z}^2 \bar{\mathfrak{z}} \det \phi, \\ b_5^0 &= \mathfrak{z}^2 \bar{\mathfrak{z}}^2 \det \phi \det \phi^*, \end{aligned} \tag{18}$$

with the abbreviations $\phi = \phi(z, z + 1), V^0 = V(\phi(z, z + 1))$ and $l = 4[\det \phi]k$ as defined above.

6 Integration of the fermions

We integrate the Fermi fields $\psi(z)$ and $\bar{\psi}(z)$ attached to sites of the lattice Λ^0 which are not sites of the block lattice Λ^1 . This yields a semi-effective Boltzmann factor $\hat{B}^1(\psi^n, \bar{\psi}^n, \phi)$ which depends on the Fermi fields on the block lattice only but will still involve the lattice gauge field ϕ on all links of the original lattice Λ^0 .

We admit Higgs fields $\phi(z + 1, z) \in GL(2, \mathbf{C})$. We will make use of the fact that for $\phi \in GL(2, \mathbf{C})$ (15) is valid and $-\epsilon = \epsilon^T = \epsilon^{-1}$. For $\phi \in SL(2, \mathbf{C})$, $\det \phi = 1$.

6.1 Basic integration formulae

We have two Grassmann variables, ψ_α , $\alpha = 1, 2$, attached to each site $z \in \Lambda^0$ and two further Grassmann variables $\bar{\psi}_\alpha$. According to the rules for Berezin integrals

$$\int d^2\psi \psi_\alpha = 0, \quad \int d^2\psi \psi_\alpha \psi_\beta = \epsilon_{\alpha\beta},$$

$$\int d^2\bar{\psi} \bar{\psi}_\alpha = 0, \quad \int d^2\bar{\psi} \bar{\psi}_\alpha \bar{\psi}_\beta = \epsilon_{\alpha\beta}. \quad (19)$$

6.2 The need for odd block lattice spacing

One might think of defining an effective Boltzmann factor on a block lattice of lattice spacing $a_1 = 2$ by integrating out the Fermi fields $\psi(z), \bar{\psi}(z)$ at even lattice sites z . However the resulting Boltzmann possesses no logarithm. To see the problem, it suffices to consider the original action (1) with Boltzmannian $B_F^0(z, z+1)e^{-V(\phi(z), \phi(z+1))}$ and with $\mathfrak{z} = \bar{\mathfrak{z}} = 1$. Expanding the exponentials $B_F(z, z+1)$ into finite sums of non-vanishing terms and applying (19) one computes $\int d^2\psi(z)d^2\bar{\psi}(z)B_F^0(z-1, z)B_F^0(z, z+1)$. One finds that it contains no term of zeroth order in the Fermi fields. Therefore it possesses no logarithm, and the effective action is not defined.

6.3 The semi-effective Boltzmannian for block size 3

The problem disappears when we take odd lattice spacing $a_1 = 3$. In this case the integration of $\psi(z+1), \bar{\psi}(z+1)$ is combined with an integration of the Fermi fields attached to site $z+2$. We start from a general Boltzmannian of the form (12) and compute the semi-effective Boltzmannian,

$$\hat{B}^1(\psi, \bar{\psi}, \phi) = \prod_{z \in \Lambda^1} \hat{B}_F^1(z, z+3),$$

$$\hat{B}_F^1(z, z+3) = \int d^2\psi(z+1)d^2\bar{\psi}(z+1)d^2\psi(z+2)$$

$$\times d^2\bar{\psi}(z+2)B^0(z, z+1)$$

$$\times B^0(z+1, z+2)B^0(z+2, z+3).$$

We adopt the same choice (7) of block spin as before, viz.

$$\phi^1(z+3, z)$$

$$= \phi(z+3, z+2)\epsilon\phi(z+1, z+2)^T\epsilon^T\phi(z+1, z).$$

We retain the notation k, L introduced before in (10) and (11).

Without loss of generality, we assume that the original action lives on the lattice Λ^0 . This can be achieved by rescaling.

The semi-effective action $\hat{B}_F^1(z, z+3)$ will have the same form (12), except that the coefficients a_i^n are replaced by coefficients $\hat{a}_i^1(z, z+3)$ which depend on the Higgs field on the fine lattice Λ^0 . More particularly, they depend on $\phi_1 = \phi(z, z+1), \phi_2 = \phi(z+1, z+2) = \phi(z+2, z+1)^*, \phi_3 = \phi(z+2, z+3)$.

Lemma 2 *Let us use the abbreviations*

$$\int \dots$$

$$= \int d^2\psi(z+1)d^2\bar{\psi}(z+1)d^2\psi(z+2)d^2\bar{\psi}(z+2) \dots$$

and

$$L_1 = L^0(z, z+1), \quad L_2 = L^0(z+1, z+2),$$

$$L_3 = L^0(z+2, z+3)$$

etc., and understand that $L^1 = L^1(z, z+3)$ etc. Then

$$\int (k\bar{k})_2 = 1, \quad (20)$$

$$\int L_1(\bar{L}k)_2L_3 = L^1, \quad (21)$$

$$\int \bar{L}_1(L\bar{k})_2\bar{L}_3 = \bar{L}^1, \quad (22)$$

$$\int k_1k_2k_3 = k^1, \quad (23)$$

$$\int \bar{k}_1\bar{k}_2\bar{k}_3 = \bar{k}^1, \quad (24)$$

$$\int (\bar{L}k)_1L_2(\bar{L}k)_3 = \bar{L}^1k^1, \quad (25)$$

$$\int (L\bar{k})_1\bar{L}_2(L\bar{k})_3 = L^1\bar{k}^1, \quad (26)$$

$$\int (k\bar{k})_1(k\bar{k})_3 = k^1\bar{k}^1, \quad (27)$$

$$\int (L\bar{L})_1(L\bar{L})_2(L\bar{L})_3 = (L^1\bar{L}^1), \quad (28)$$

and all other integrals of the form $\int X_1Y_2Z_3$ vanish, for $X, Y, Z = L, \bar{L}, k, \bar{k}, \bar{L}k, \dots$

Proof. Using the rules of Berezin integration, one notices that only those factors $X_1Y_2Z_3$ do not integrate to zero for which there are exactly two factors $\psi.(z+1)$, two factors $\bar{\psi}.(z+1)$, two factors $\psi.(z+2)$ and two factors $\bar{\psi}.(z+2)$. Computing their integrals according to the rules of Sect. 6.1 furnishes the above result.

Using Lemma 2, one can compute

$$\int \hat{B}^0(z, z+1)\hat{B}^0(z+1, z+2)\hat{B}^0(z+2, z+3)$$

as a sum of terms $X_i^1 = 1, L^1, \bar{L}^1, \dots$ multiplied with coefficients $\hat{a}_i^1(z, z+3)$. We write these coefficients in the form

$$\hat{a}_i^1(z, z+3) = \sum_{jkl} d^i_{jkl} a_j(z, z+1) \bar{a}_k(z+1, z+2)$$

$$\times a_l(z+2, z+3). \quad (29)$$

We use again the notation $a_j = \bar{a}_j$. With some foresight we chose to define d^i_{jkl} so that the second factor $a_i(z+1, z+2)$ appears complex conjugated. This is merely a convention, since $\sum_{jkl} d^i_{jkl} a_j \bar{a}_k a_l = \sum_{jkl} d^i_{j\bar{k}l} a_j a_k a_l$.

With this convention one finds that the non-vanishing coefficients d^i_{jkl} are equal to 1, and they are as listed at the end of Corollary 1. We note that

$$\begin{aligned} \bar{a}_j(z+1, z+2) &= \bar{a}_j[\phi(z+1, z+2)] \\ &= a_j[\phi(z+2, z+1)]. \end{aligned} \tag{30}$$

6.4 The semi-effective Boltzmannian for $H = SL(2, \mathbf{C})$

Here we specialize to Higgs fields $\phi \in SL(2, \mathbf{C})$. In this case, both the initial values (18) and the recursion relations (29) simplify, because $\det \phi = 1$.

Proposition 1 *Consider the model with holonomy group $H = SL(2, \mathbf{C})$. Starting from the initial action (1), the effective action after the first, and therefore after every RG-step, retains the form (1). We have*

$$\begin{aligned} S^1(\psi^1, \bar{\psi}^1, \phi^1) &= \sum_{z \in A^1} S^1(z, z+3), \\ S^1(z, z+3) &= \mathfrak{z}^1 \bar{\psi}^1(z) \phi^1(z, z+3) \psi^1(z+3) \\ &\quad + \bar{\mathfrak{z}}^1 \bar{\psi}^1(z+3) \phi^1(z+3, z) \psi^1(z) \\ &\quad - V^1(\phi^1(z, z+3)). \end{aligned}$$

In particular the semi-effective Boltzmannian for $H = SL(2, \mathbf{C})$ after one RG-step has the form

$$\begin{aligned} \hat{B}^1(z, z+3) &= (\mathfrak{z}\bar{\mathfrak{z}})^2 e^{S^1_{\mathbb{F}}(z, z+3)} \\ &\quad \times e^{-V(\phi(z, z+1)) - V(\phi(z+1, z+2)) - V(\phi(z+2, z+3))} \\ S^1_{\mathbb{F}}(z, z+3) &= \mathfrak{z}^1 \bar{\psi}^1(z) \phi^1(z, z+3) \psi^1(z+3) \\ &\quad + \bar{\mathfrak{z}}^1 \bar{\psi}^1(z+3) \phi^1(z+3, z) \psi^1(z). \end{aligned}$$

with $\mathfrak{z}^1 = -(\mathfrak{z})^2 \bar{\mathfrak{z}}^{-1}$ and effective Higgs potential V^1 as described below. $a_i^1 = (\mathfrak{z})^2 (\bar{\mathfrak{z}})^2 b_i^1 e^{-V^1}$ and b_i^1 is as in (18) after having replaced \mathfrak{z} by \mathfrak{z}^1 with $\det \phi = 1$.

Proof. Because \mathfrak{z} are constants and $\det \phi = 1$, the starting coefficients a_i^0 , (17), which determine the original Boltzmannian (16), become ϕ independent except for overall factors $e^{-V^0(\phi(z, z+1))}$ common to all of them which may be pulled out. It follows from (29) that the coefficients \hat{a}_i^1 have the same form as the coefficients in the original Boltzmannian (16) except for the common overall-factor

$$(\mathfrak{z}\bar{\mathfrak{z}})^2 e^{-V^0(\phi(z, z+1)) - V^0(\phi(z+1, z+2)) - V^0(\phi(z+2, z+3))}$$

and replacement of \mathfrak{z} by \mathfrak{z}^1 as defined in the proposition, the factor $(\mathfrak{z}\bar{\mathfrak{z}})^2$ remains in the Boltzmannian. Note that $(\mathfrak{z}^1 \bar{\mathfrak{z}}^1)^2 = (\mathfrak{z}\bar{\mathfrak{z}})^2$.

Since the original Boltzmannian (16) came from the action (1), the same will be true with appropriate adjustments for the semi-effective Boltzmannian, as stated in the proposition.

7 Consequences of gauge invariance

Here we derive the consequences of $SU(2)$ gauge invariance, including (33).

Under $SU(2)$ gauge transformations, $\phi(z, z+1) \mapsto u(z)\phi(z, z+1)u(z+1)^{-1}$. Since $u(z)$ and $u(z+1)$ are both arbitrary elements of $SU(2)$, gauge invariance of V implies that $V(\Phi) = V(u_1\Phi u_2)$. It follows from the parametrization (32) of $A \in SL(2, \mathbf{C})$ that $V(\zeta A) = V(\zeta d(\eta))$. The remaining statement of (33) is contained in the following lemma. For later convenience we state it for general functions V on H .

Lemma 3 (Properties of $SU(2)$ -bi-invariant functions) *Suppose $SU(2) \subset H \subseteq GL(2, \mathbf{C})$. Let V be a function of $\Phi \in H$, so that $\Phi = \zeta A$ with $0 \neq \zeta \in \mathbf{C}$ and $A \in SL(2, \mathbf{C})$ which is $SU(2)$ -bi-invariant in the sense that $V(\Phi) = V(u_1\Phi u_2)$ for all $u_1, u_2 \in SU(2)$. Then*

$$V(\zeta A) = V(\zeta A^{-1}) = V(\zeta \epsilon A^T \epsilon^T).$$

In particular, for $H = SL(2, \mathbf{C})$ and $d(\eta) = \text{diag}(e^{-\eta/2}, e^{\eta/2})$

$$V(d(\eta)) = V(d(-\eta)).$$

Proof. Every element $A \in SL(2, \mathbf{C})$ may be decomposed as $A = u_1 d(\eta) u_2$ with $u_1, u_2 \in SU(2)$, $\eta \in \mathbf{R}$, and $d(\eta)$ as in (32).

There exists a $w \in SU(2)$ such that $wd(\eta)w^{-1} = d(\eta)^{-1} = d(-\eta)$, viz.

$$w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{31}$$

It follows that

$$\begin{aligned} V(\zeta A) &= V(\zeta d(\eta)) = V(\zeta wd(\eta)w^{-1}) = V(\zeta d(\eta)^{-1}) \\ &= V(\zeta u_2^{-1} d(\eta)^{-1} u_1^{-1}) = V(\zeta A^{-1}). \end{aligned}$$

The last equation follows from (15) with A substituted for ϕ , since $\det A = 1$.

8 Integration of the Higgs field for $H = SL(2, \mathbf{C})$

After the integration of the Fermi fields, we obtained a semi-effective Boltzmann factor \hat{B}^1 which still depends on the Higgs fields ϕ on the links of the original lattice. One sees from the result Proposition 1 of Sect. 6.4 for $H = SL(2, \mathbf{C})$,

that the only dependence of \hat{B}^1 on ϕ other than the dependence on the block spin ϕ^1 is in the factors $e^{-V(\phi(z,z+1))}$. Therefore it only remains to compute the flow of the Higgs potential. With the choice of block spin (7), the flow of the effective potential is determined by the recursion relation

$$e^{-V^{n+1}(\Phi)} = \int d\phi_1 d\phi_2 d\phi_3 \delta(\phi_1 \epsilon \phi_2^T \epsilon^T \phi_3 \Phi^{-1}) \times e^{-V^n(\phi_1) - V^n(\phi_2) - V^n(\phi_3)}.$$

For $\phi \in SL(2, \mathbf{C})$, $\epsilon \phi_2^T \epsilon^T = \phi_2^{-1}$ by (15).

It is easy to see that the bi-invariance property of V passes through the recursion relation. Therefore we have Lemma 3 available, with $\zeta = 1$, permitting a change of variable so that

$$e^{-V^{n+1}(\Phi)} = \int d\phi_1 d\phi_2 d\phi_3 \delta(\phi_1 \phi_2 \phi_3 \Phi^{-1}) \times e^{-V^n(\phi_1) - V^n(\phi_2) - V^n(\phi_3)}.$$

Lemma 4 (Convolution) *Under harmonic analysis on $SL(2, \mathbf{C})$, convolution goes into products. Thus if the square integrable functions f_i admit an expansion*

$$f_i(A) = \int d\chi \text{tr}_\chi \left(\tilde{F}_i(\chi) T^\chi(A) \right)$$

for $i = 1, \dots, N$, then

$$\begin{aligned} & \int dA_1 \dots dA_N f_1(A_1) \dots f_n(A_n) \delta(A_1 \dots A_N A^{-1}) \\ &= \int d\chi \text{tr}_\chi \left(\tilde{F}_1(\chi) \dots \tilde{F}_N(\chi) T^\chi(A) \right). \end{aligned}$$

Herein, $d\chi$ is the Plancherel measure on $SL(2, \mathbf{C})$, and $T^\chi(A)$ are the representation operators for representations of the unitary principal series (see Appendix A).

This is well known and generalizes to the other groups H .

Every element A of $SL(2, \mathbf{C})$ may be parametrized as

$$A = u_1 d(\eta) u_2, \text{ with } u_1, u_2 \in SU(2),$$

$$d(\eta) = \begin{pmatrix} e^{-\eta/2} & 0 \\ 0 & e^{\eta/2} \end{pmatrix}. \tag{32}$$

It follows from gauge invariance that

$$V(A) = V(d(\eta)) = V(A^{-1}), \tag{33}$$

so that the Higgs potential is a function of a single variable η . We describe the flow of the effective Higgs potential. It is derived by harmonic analysis on the non-compact group $SL(2, \mathbf{C})$ and will involve the principal series representation function $P^\rho(A) = 2 \sin(\frac{1}{2}\eta\rho) / \rho \sinh \eta$ for real ρ .

Theorem 2 (Flow of the Higgs potential for $H = SL(2, \mathbf{C})$) *Given the initial Higgs potential $V(\phi)$, define*

$$b(\rho) = \int \sinh^2 \eta d\eta e^{-V(d(\eta))} \frac{2 \sin(\frac{1}{2}\eta\rho)}{\rho \sinh \eta}$$

for real ρ . Then after n RG-steps,

$$e^{-V^n(d(\eta))} = \frac{1}{8\pi} \int_{-\infty}^{\infty} \rho^2 d\rho [\mathfrak{z}\bar{\mathfrak{z}}]^{2n} [b(\rho)]^N \frac{2 \sin(\frac{1}{2}\eta\rho)}{\rho \sinh \eta},$$

with $N = 3^n$.

The factor $[\mathfrak{z}\bar{\mathfrak{z}}]^{2n}$ shows the essential role of the fermions.

Because of the recursions relation $\mathfrak{z}^1 = (\mathfrak{z})^2(\bar{\mathfrak{z}})^{-1}$ we have $(\mathfrak{z}^1)^2(\bar{\mathfrak{z}}^1)^2 = (\mathfrak{z})^2(\bar{\mathfrak{z}})^2$.

The result of Theorem 2 is now an immediate consequence of Lemma 4 and of Lemma 3.

The $SL(2, \mathbf{C})$ -model is fairly trivial. The integration of the ‘‘high frequency components’’ of the fermions does not affect the flow of the Higgs potential. And the integration of the ‘‘high frequency component’’ of the Higgs field ϕ does not produce 4-fermion interactions, 6-fermion interactions, or higher.

This is easy to understand. Because $\psi(z)$ and $\bar{\psi}(z)$ are independent variables, they may be transformed independently. The fermionic part of the action will be invariant under the $SL(2, \mathbf{C})$ gauge transformations,

$$\begin{aligned} \psi(z) &\mapsto A(z)\psi(z), \\ \bar{\psi}(z) &\mapsto \bar{\psi}(z)A^*(z), \\ \phi(z+1, z) &\mapsto A(z+1)^{* -1} \phi(z+1, z)A(z)^{-1}. \end{aligned}$$

This can be used to transform the parallel transporters in the fermionic action away. Only the Higgs potential is not $SL(2, \mathbf{C})$ -invariant.

9 Effective Boltzmannian for general H

Here we derive the recursion relations for the coefficients a_i^n of the Boltzmann factor for general H , starting from an arbitrary action of the form (12).

Without loss of generality, we assume that the original action lives on the lattice Λ^0 . This can be achieved by rescaling.

Given the semi-effective Boltzmannian of Sect. 6.3, we need to perform the integration over the Higgs fields, keeping the block spin fixed. Switching to variables $\phi_1 = \phi(z, z+1)$, $\phi_2 = \phi(z+2, z+1)$ (sic!), $\phi_3 = \phi(z+2, z+3)$ we have to compute the coefficients

$$a_i^1[\Phi] = \int d\phi_1 d\phi_2 d\phi_3 \hat{a}_i^1(z, z+3) \delta(\phi_1 \epsilon \phi_2^T \epsilon^T \phi_3 \Phi^{-1}).$$

Inserting (29), and using (30) and our notational conventions, (9) takes the form

$$\begin{aligned} a_i^1[\Phi] &= \sum_{jkl} d^i_{jkl} \int d\phi_1 d\phi_2 d\phi_3 a_j[\phi_1] \bar{a}_k[\phi_2] a_l[\phi_3] \\ &\quad \times \delta(\phi_1 \epsilon \phi_2^T \epsilon^T \phi_3 \Phi^{-1}). \end{aligned}$$

The coefficients a_i are $SU(2)$ -bi-invariant functions. Therefore Lemma 3 is applicable to them. We use this to make the change of variable $\phi'_2 = \epsilon \phi^T \epsilon^T$ to obtain

$$a_i^1[\Phi] = \sum_{jkl} d^i_{jkl} \int d\phi_1 d\phi'_2 d\phi_3 a_j[\phi_1] a_k[\phi'_2] a_l[\phi_3]$$

$$\begin{aligned} & \times \delta(\phi_1 \phi_2' \phi_3 \Phi^{-1}) \\ & = \sum_{jkl} d^i_{jkl} a_j * a_k * a_l[\Phi]. \end{aligned}$$

Remembering the form of d^i_{jkl} , this demonstrates Theorem 1. The explicit coefficients are given in Theorem 1. Finally we present the solutions of the recursion relations in terms of the initial values a_j^0 .

Theorem 3 (Solution of the flow equations) *For temporary use we introduce the notation $(A)^{*N} = A * A * \dots * A$ (N factors) for the N -fold convolution product. In this notation*

$$\begin{aligned} a_0^n[\Phi] &= a_0^0 * (\bar{a}_5^0 * a_0^0)^{*N}[\Phi], \\ a_1^n[\Phi] &= a_1^0 * (\bar{a}_4^0 * a_1^0)^{*N}[\Phi], \\ a_2^n[\Phi] &= a_2^0 * (\bar{a}_2^0 * a_2^0)^{*N}[\Phi], \\ a_3^n[\Phi] &= a_3^0 * (\bar{a}_3^0 * a_3^0)^{*N}[\Phi], \\ a_4^n[\Phi] &= a_4^0 * (\bar{a}_1^0 * a_4^0)^{*N}[\Phi], \\ a_5^n[\Phi] &= a_5^0 * (\bar{a}_0^0 * a_5^0)^{*N}[\Phi], \end{aligned}$$

with $N = \frac{1}{2}(3^n - 1)$.

The Fourier coefficients $\tilde{a}_j^n(\Xi)$ obey the same relations, with ordinary products in place of convolution products.

The proof is immediate from Theorem 1. To use the recursion relations, we need the starting values of the coefficients $a_i(z, z+1)$, and their complex conjugates, that were already given in (18).

10 Taking the logarithm

By straightforward computation one deduces

Proposition 2 (Effective action from effective Boltzmannian) *Retaining the assumption $L^2 = -2[\det \phi]k$, the coefficients c_i^n in the effective action (9) are obtained from the coefficients a_j^n in the effective Boltzmannian (12) as follows:*

$$\begin{aligned} c_0^n &= \ln(a_0^n), \\ c_1^n &= b_1, \\ c_2^n &= b_2 + b_1^2[\det \phi], \\ c_3^n &= b_3 - b_1 \bar{b}_1, \\ c_4^n &= \bar{b}_4 - \bar{b}_1 b_2 + 2b_1 b_3[\det \phi] - 2b_1^2 \bar{b}_1[\det \phi], \\ c_5^n &= b_5 - b_2 \bar{b}_2 + \left(2b_1 \bar{b}_4 - \frac{4}{3} b_1^2 \bar{b}_2\right) [\det \phi] \\ & \quad + \left(2\bar{b}_1 b_4 - \frac{4}{3} \bar{b}_1^2 b_2\right) [\det \phi^*] \\ & \quad + (8b_1 \bar{b}_1 b_3 - 2b_3^2 - 6b_1^2 \bar{b}_1^2) [\det \phi][\det \phi^*], \end{aligned}$$

where $b_i = \frac{a_i^n}{a_0^n}$ and \bar{c}_2^n, \bar{c}_4^n are obtained by complex conjugation.

11 Asymptotic behavior after many RG-steps

We assume that $\mathfrak{z} = \bar{\mathfrak{z}} = \pm 1$.

Theorem 3 gives the behavior of the coefficients of the effective Boltzmannian after n steps in the form of a convolution product of $2N + 1 = 3^n$ factors $a_i^0[\Phi]$ (with $N = \frac{1}{2}(3^n - 1)$).

The central limit theorem gives the asymptotic behavior of such convolution products in the limit $N \mapsto \infty$, provided that the coefficients $a_j^0[\Phi]d\Phi$ are normalizable measures, or negatives of such. Assuming $\mathfrak{z} = \bar{\mathfrak{z}} = \pm 1$, this is always the case for a_0^0 and for a_5^0 , which enter the formula for the exponentiated Higgs potential $a_0^n = e^{-V^n}$, and this is also the case in all cases except for $G = SU(2), H = GL(2, \mathbf{C})$ for the other coefficients.

We need a central limit theorem for G -bi-covariant functions on the non-compact groups H . The principle from which it flows is as follows. Under appropriate conditions (“good functions f ”) the Fourier coefficient $\tilde{f}(\Xi)$ for normalized measures $f d\Phi$ has an isolated absolute maximum at a special representation Ξ_0 . The prototype of such a result, which stands behind the central limit theorem on \mathbf{R} , is Lemma 5 below. As a result, $\tilde{f}(\Xi)^N$, which is the Fourier coefficient of $f[\Phi]^{*N}$, is well approximated by a Gaussian, and therefore $f[\Phi]^{*N}$ can be accurately computed as inverse transform of a Gaussian. The non-trivial part of the proof of a central limit theorem is the control of errors. To give a rigorous proof is outside the scope of this paper. We will only present a heuristic derivation, following the above outline.

In the following subsections, we consider the exponentiated Higgs potential. The other coefficients in the effective Boltzmannian can be treated in the same way, provided the coefficients which enter into the recursion relation are either positive or negative definite. This is true in all cases except for $G = SU(2), H = GL(2, \mathbf{C})$.

Lemma 5 (Fourier transform of normalized measures on \mathbf{R}) *Let f be an integrable and square integrable positive definite function on the real line with $\int f(x)dx = 1$. Then the modulus of its Fourier transform $\tilde{f}(k) = \int e^{ikx} f(x)dx$ has a unique absolute maximum at $k = 0$ with $\tilde{f}(0) = 1$.*

Proof. From the definition of the Fourier transform and from $\int f dx = 1$ it follows that

$$|\tilde{f}(k)|^2 = |\tilde{f}(0)|^2 - \int dx dy [1 - \cos k(x - y)] f(x) f(y).$$

The second term is negative semi-definite and for any $k \neq 0$ it can only vanish if f is either a δ -function or a series of δ -functions with supports whose distance is an integer multiple of $2\pi/k$. This is excluded by the assumption that f is square integrable. Therefore the only absolute maximum is at $k = 0$. The assertion $\tilde{f}(0) = 1$ is a restatement of $\int f(x)dx = 1$.

11.1 The case $H = SL(2, \mathbf{C})$

We consider the $SU(2)$ -bi-invariant function

$$f(\Phi) = e^{-V(\Phi)} = f(d(\eta)),$$

which is positive definite. By our assumptions, the additive constant in V can be so chosen that $f(\Phi)d\Phi$ is a normalized measure on $SL(2, \mathbf{C})$.

The Fourier coefficient is

$$\begin{aligned} \tilde{f}(\rho) &= \int d\Phi f(\Phi) P^\rho(\Phi^{-1}) \\ &= \int \sinh^2 \eta d\eta f(d(\eta)) \frac{2 \sin \frac{1}{2} \eta \rho}{\rho \sinh \eta}. \end{aligned}$$

It is real, obeys $\tilde{f}(\rho) = \tilde{f}(-\rho)$ and has an isolated absolute maximum 1 at $\rho = 0$. This follows from the fact that $\lim_{\rho \rightarrow 0} \frac{2 \sin \frac{1}{2} \eta \rho}{\rho \sinh \eta} = 1$, while $|\frac{2 \sin \frac{1}{2} \eta \rho}{\rho \sinh \eta}| \leq 1$ and $|\eta^{-1} \sinh \eta| > 1$ for $\eta \neq 0$.

It follows from the unique maximum property that

$$\begin{aligned} \tilde{f}(\rho) &= e^{-\frac{b}{2} \rho^2} + \dots, \\ b &= -\tilde{f}''(0) = \frac{1}{12} \int \sinh^2 \eta d\eta f(d(\eta)) \eta^2. \end{aligned}$$

Now set $B = Nb$, $N = 3^n$. To obtain the asymptotic behavior of the exponentiated effective Higgs potential, we have to back-transform $\tilde{f}(\eta)^N = e^{-\frac{B}{2} \rho^2}$, i.e. compute

$$e^{-V^n(d(\eta))} \sim \int \frac{\rho^2 d\rho}{8\pi} \frac{2 \sin \frac{1}{2} \eta \rho}{\rho \sinh \eta} e^{-\frac{B}{2} \rho^2}.$$

The integral evaluates to

$$e^{-V^n(d(\eta))} \sim \frac{1}{(2\pi B)^{\frac{3}{2}}} \frac{\eta}{4 \sinh \eta} e^{-\frac{1}{8B} \eta^2}.$$

This completes the derivation of the asymptotic behavior of the effective Higgs potential (14) for $H = SL(2, \mathbf{C})$.

11.2 The group $\mathbf{R}_+SU(2)$

We specify the representations by $s = i\sigma$ imaginary. $SU(2)$ -bi-covariant functions f of $ru \in \mathbf{R}_+SU(2)$ depend only on r , and the harmonic analysis amounts to a Mellin transformation (cf. Appendix A), which in turn is related to a standard Fourier transformation,

$$\tilde{f}(i\sigma) = \int_0^\infty dr r^{i\sigma-1} f(r) = \int_{-\infty}^\infty dx e^{i\sigma x} f(e^x).$$

Let f be positive definite and a good function. Then, by Lemma 5, $|\tilde{f}|$ has a maximum at $\sigma = 0$ and

$$\tilde{f}(i\sigma) = e^{-\frac{b}{2} \sigma^2 + ia\sigma} + \dots$$

if f is normalized appropriately. The coefficients are

$$\begin{aligned} a &= -i \frac{\partial}{\partial \sigma} f(i\sigma) \Big|_{\sigma=0} = \int \frac{dr}{r} \ln r f(r), \\ b &= -\frac{\partial^2}{\partial \sigma^2} f(i\sigma) \Big|_{\sigma=0} = \int \frac{dr}{r} \ln^2 r f(r). \end{aligned}$$

The exponentiated effective Higgs potential is given by the following convolution product on $\mathbf{R}_+SU(2)$:

$$e^{-V^n} = a_0^{*N+1} * a_5^{*N},$$

with $N = \frac{1}{2}(3^n - 1)$, $a_i = a_i^0$, $a_0 = e^{-V}$ and $a_5 = |\det \Phi|^2 e^{-V}$.

We choose the constant in V so that $a_0 * a_5$ is a normalized measure on H and define $\mathfrak{N} = \int d\phi e^{-V(\phi)}$. We have to evaluate

$$e^{-V^n(r)} \sim \int_{-\infty}^\infty d\sigma r^{i\sigma} \tilde{a}_0(i\sigma) (\widetilde{a_0 * a_5})^N(i\sigma).$$

$a_0 * a_5 d\Phi$ is a convolution of normalizable measures and therefore a normalizable measure itself. Since a_0 is positive definite, $a_0 * a_5$ is positive definite, implying the unique maximum property. As a consequence, there are constants $b > 0$, $b' > 0$ and a, a' such that $\tilde{a}_0(i\sigma) = \mathfrak{N} e^{-\frac{b'}{2} \sigma^2 + i\sigma a'}$, $(\widetilde{a_0 * a_5})(i\sigma) = e^{-\frac{b}{2} \sigma^2 + i\sigma a}$. Upon insertion of this and after a change of variables to $x = \ln r$, the integral becomes Gaussian and evaluates to

$$e^{-V^n(r)} \sim \frac{\mathfrak{N}}{(2\pi B)^{\frac{1}{2}}} \exp\left(-\frac{1}{2B} \ln^2(re^A)\right),$$

with $B = b' + Nb$, $A = a' + Na$. Their explicit value is determined by the starting Higgs potential V and is obtained by applying (34).

11.3 The group $GL(2, \mathbf{C})$

This combines aspects of the previous two cases. It is worked out in the same way, using the results of Appendix A. The maximum of $\tilde{f}(\Xi)$, $\Xi = (l, s, \rho)$ is at $l = 0, s = 0, \rho = 0$. The dominance of $l = 0$ gives independence of the asymptotic behavior of $\exp(-V^n(\zeta d(\eta)))$ of the phase of ζ . The group $\mathbf{R}_+SL(2, \mathbf{C})$ is treated in the same way. Here, the quantum number l is absent.

12 Model with gauge group $U(2)$

The results can be extended to the model with gauge group $U(2)$ and $H = GL(2, \mathbf{C})$. To do so, we have to investigate the consequences of an additional local $U(1)$ -invariance. Under a $U(1)$ -transformation,

$$\begin{aligned} \psi(z) &\mapsto e^{-i\vartheta(z)} \psi(z), \\ \bar{\psi}(z+1) &\mapsto e^{i\vartheta(z+1)} \bar{\psi}(z+1), \\ \phi(z, z+1) &\mapsto e^{-i\vartheta(z)} \phi(z, z+1) e^{i\vartheta(z+1)}. \end{aligned}$$

Local $U(1)$ -invariance implies that the Higgs-potential $V(\zeta A)$ is independent of the phase of ζ .

The block spin (7) is not $U(1)$ -covariant, and must be changed by a factor which involves the phase of $\det \phi^n$ to

$$\phi^{n+1}(z, z+3a_n) = e^{i \arg \det \phi^n(z+a_n, z+2a_n)} \phi^n(z, z+a_n) \epsilon$$

$$\begin{aligned} &\times \phi^n(z + 2a_n, z + a_n)^T \epsilon^T \\ &\times \phi^n(z + 2a_n, z + 3a_n). \end{aligned} \tag{34}$$

It is appropriate to change also the definition (11) of k to

$$k^n(z, z + a_n) = \frac{1}{4} e^{i \arg \det \phi^n(z, z + a_n)} \bar{\psi} \epsilon \bar{\psi}(z) \times \psi \epsilon \psi(z + a_n).$$

When these changes are made, the master integration formulas (20)–(28) remain valid as they stand. Therefore the recursion relations Theorem 1 for the coefficients a_j^n remain literally valid, and also Corollary 1 remains valid with the understanding that harmonic expansion of $U(2)$ -bi-covariant functions on $GL(2, \mathbb{C})$ enters.

The initial values (18) change as a consequence of the changed definition of k . In the formulas (18), $\det \phi$ and $\det \phi^*$ have to be replaced by $|\det \phi|$.

As a consequence, given that $\mathfrak{z} = \bar{\mathfrak{z}} = \pm 1$, all the quantities $a_j^0[\phi]d\phi$ are normalizable measures, or negatives of normalizable measure. It follows from this in the same way as for the other models with this property that the model has a RG-flow from the UV fixed point to a IR fixed point. The asymptotic behavior of the Higgs potential is the same as in the $G = SU(2), H = GL(2, \mathbb{C})$ model.

13 Outlook

We are interested in the question how Higgs fields come into an effective field theory at intermediate scales, and what their geometrical interpretation is. A renormalization group mechanism was proposed. The one-dimensional model studied in this paper illustrates the mechanism in part. Let us discuss what has to be added and how it involves the conventional four dimensions which have been ignored in the model. We call these the perpendicular directions.

In the one-dimensional model, the ordinary gauge theory is a fixed point. We need a theory in which there are some, albeit arbitrarily small, fluctuations of the length of ϕ to start a non-trivial flow.

When the perpendicular directions are taken into account, two things are expected to happen.

(1) Coarsening in the perpendicular directions, which is involved in the construction of a block spin ϕ from unitary parallel transporters u (on the links along the extra direction), will produce ϕ in the linear span, of G . Therefore they have a fluctuating length of some kind. (The non-invertible ϕ will have measure zero.)

(2) At some scale we will be prevented from reducing to a ordinary gauge theory by integrating out the length of ϕ (more precisely the self-adjoint factor p in the polar decomposition $\phi = pu, u$ unitary), because otherwise the effective theory will become non-local in the perpendicular directions. This will happen when the mass m determined by the curvature at the minima of the Higgs potential falls below the cutoff scale M .

In conclusion, there will be a domain of scales where the effective theory needs a Higgs field for its locality. In principle, one could also study the mechanisms envisaged here by numerical means.

Let us now turn to the quark sector of the standard model. We give a brief review of results of Olschewski and Angermann taken from [15].

We imagine that all three generations of left handed quarks belong to one irreducible representation space of the holonomy group H .

Ignoring color, the representation space of H in the standard model is a six-dimensional space $V_L \times \mathbb{C}^3$, where V_L is the two-dimensional representation space of the left handed gauge group $G_L = SU(2) \times U(1)$. Similarly, the right handed quarks belong to an isomorphic six-dimensional representation space $(V_R \oplus \tilde{V}_R) \times \mathbb{C}^3$ of H , where V_R and \tilde{V}_R are one-dimensional representation spaces of the right handed gauge group $G_R = U(1)$ (with $Y = +2/3$ and $Y = -1/3$, respectively) The Higgs fields in our sense map between them:

$$\Phi : (V_R \oplus \tilde{V}_R) \times \mathbb{C}^3 \mapsto V_L \times \mathbb{C}^3. \tag{35}$$

In a defect model [15], the Higgs fields Φ and their adjoints Φ^* are pictured as non-unitary parallel transporters across a defect which has some extension in a fifth direction; cf. Fig. 1 below. The left handed quarks live below the defect, and the right handed quarks live above.

There will be a Higgs potential $V(\Phi)$. We imagine that the Higgs potential is to be determined as the result of a renormalization group flow, similarly as in the very simplified model of this paper. Unfortunately we are unable to say, for now, where the flow starts. In the simplified model, the asymptotic behavior of the Higgs model is quite insensitive to where the flow starts. This reflects universality.

If the Higgs potential could be computed, the quark masses could be determined, as explained in Sect. 3. The value of Φ at the minimum of the Higgs potential will contain the information that goes into the quark masses.

Writing φ for the conventional Higgs doublet,

$$\varphi = \begin{pmatrix} \varphi_0 \\ \varphi_+ \end{pmatrix}, \quad \tilde{\varphi} = \begin{pmatrix} -\bar{\varphi}_+ \\ \bar{\varphi}_0 \end{pmatrix}, \quad \begin{aligned} \varphi &: V_R \rightarrow V_L \\ \tilde{\varphi} &: \tilde{V}_R \rightarrow V_L \end{aligned}, \tag{36}$$

the Higgs parallel transporter Φ through the defect will have the form

$$\Phi = (\varphi M_U, \tilde{\varphi} M_D) : (V_R \oplus \tilde{V}_R) \times \mathbb{C}^3 \rightarrow V_L \times \mathbb{C}^3. \tag{37}$$

Neglecting fluctuations of the Higgs field Φ around the minimum of the Higgs potential, M_U and M_D will be constant 3×3 matrices

$$M_U, M_D : \mathbb{C}^3 \mapsto \mathbb{C}^3. \tag{38}$$

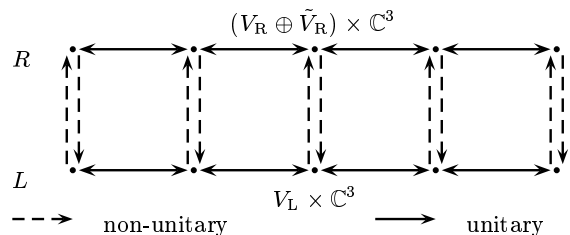


Fig. 1. Bi-layered membrane

The matrices M_U, M_D can be diagonalized through a bi-unitary transformation,

$$M_U = A_L^* m_U A_R, \quad m_U = \text{diag}(m_u, m_c, m_t), \quad (39)$$

$$M_D = B_L^* m_D B_R, \quad m_D = \text{diag}(m_d, m_s, m_b). \quad (40)$$

Herein, m_u, \dots are the quark masses, and A_L, B_L, A_R, B_R are unitary. By a basis change, A_R, B_R can be transformed away. The Kobayashi–Maskawa matrix is $C = A_L B_L^*$.

Consider now the elements of the holonomy group H_{tot} of the whole theory, including the boundaries of the defect which were not modeled in our one-dimensional theory.

The elements of H_{tot} are the parallel transporters along paths which may pass through the defect, possibly several times forth and back, and their inverses. The parallel transporters along pieces of path above the defect will be of the form $U_R \mathbf{1}$, where $\mathbf{1}$ is the 3×3 unit matrix, and U_R are diagonal 2×2 matrices whose non-vanishing entries are representation operators of $U(1)$, and the parallel transporters along pieces of path below the defect will be unitary matrices $U_L \mathbf{1}$ with

$$U_L = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \in SU(2) \times U(1). \quad (41)$$

Going to unitary gauge $\varphi_0 = \rho, \varphi_+ = 0$, the parallel transporter along a closed path which passes once across the defect, forth and back, will be of the form

$$U_R \Phi^* U_L \Phi = U_R \begin{pmatrix} m_U & 0 \\ 0 & m_D \end{pmatrix} \times \begin{pmatrix} U_{11} \mathbf{1} & C U_{12} \\ C^* U_{21} & U_{22} \mathbf{1} \end{pmatrix} \begin{pmatrix} m_U & 0 \\ 0 & m_D \end{pmatrix} \rho^2, \quad (42)$$

where C is again the Kobayashi–Maskawa matrix. The difference of the quark masses is responsible for the fact that the holonomy group H is larger than the unitary gauge group. It involves non-unitary matrices and is therefore non-compact.

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Appendix A: Harmonic analysis

Here we present details of the expansion of the $SU(2)$ -bi-invariant functions $f(\Phi)$ on H in the spherical functions $P^\Xi(\Phi)$. They are associated with unitary representations Ξ of H . The general expansion formula reads

$$f[\Phi] = \int d\Xi \tilde{f}(\Xi) P^\Xi(\Phi),$$

where $d\Xi$ is the restriction of the Plancherel measure on H to representations which contribute to the expansion of $SU(2)$ -bi-invariant functions.

Denoting by $\bar{\Xi}$ the complex conjugate representation, the complex conjugate function \bar{f} expands according to

$$\bar{f}[\Phi] = \int d\bar{\Xi} \tilde{f}(\bar{\Xi}) P^{\bar{\Xi}}(\Phi).$$

Harmonic analysis on $SL(2, \mathbf{C})$

We use the parametrization (32) of elements $SL(2, \mathbf{C})$ in terms of variables $u_1, u_2 \in SU(2)$ and $\eta \in \mathbf{R}$.

Integration with the Haar measure in these variables can be written as

$$\int d\phi f(\phi) = \int du_1 du_2 \sinh^2 \eta d\eta f(u_1 d(\eta) u_2), \quad (43)$$

where du is the normalized Haar measure on the group $SU(2)$. The formula involves a redundant integration over the commutant of $d(\eta)$ in $SU(2)$.

We wish to use harmonic analysis on the group $SL(2, \mathbf{C})$ to determine the RG-flow of functions f like the exponentiated Higgs potential V in case $H = SL(2, \mathbf{C})$. They are functions on $H = SL(2, \mathbf{C})$. We assume that $V(d(\eta))$ increases sufficiently fast at large $|\eta|$ to ensure that these functions are square integrable on the group. This ensures [17, 18] that it can be expanded in representation functions of the principal series of unitary representations of $SL(2, \mathbf{C})$.

The representations of the Lorentz group $SO(3, 1)$ in the principal series are labeled by $\chi = (l, \rho)$, where l labels a representation of $SO(2)$, $l = 0, \pm 1, \pm 2, \dots$ and ρ is a real number. $SL(2, \mathbf{C})$ is the twofold covering of $SO(3, 1)$. Its principal series representations are labeled in the same way, except that $l = 0, \pm \frac{1}{2}, \pm 1, \dots$. The representation operators $T^\chi(A)$ act on Hilbert spaces \mathcal{H}^χ . The expansion of a square integrable function on the group reads³

$$f(A) = \int d\chi \text{tr}_\chi \left(\tilde{F}(\chi) T^\chi(A) \right),$$

where $d\chi$ is the Plancherel measure on $SL(2, \mathbf{C})$

$$\int d\chi \dots = \sum_l \int_{-\infty}^{\infty} (l^2 + \rho^2) \frac{d\rho}{8\pi} \dots,$$

tr_χ is the trace over \mathcal{H}^χ , and $\tilde{F}(\chi)$ is an operator in \mathcal{H}^χ . The inversion formula reads

$$\tilde{F}(\chi) = \int_{SL(2, \mathbf{C})} d\phi f(\phi) T^\chi(\phi^{-1}).$$

We apply this to the functions $f(A)$ which are bi-invariant in the sense that $f(u_1 A u_2) = f(A)$ for all $u_1, u_2 \in SU(2)$. A great simplification results. The representation space \mathcal{H}^χ decomposes into irreducible representations of $SU(2)$ as follows [19, 20]. Every irreducible representation of $SU(2)$ occurs at most once, and the $2k + 1$ -dimensional representation occurs if $k \geq |l|$, $k - l$ integer. In particular, the

³ $l = \frac{1}{2} m$ in Rühl's notation [19].

trivial representation of $SU(2)$ occurs only for $l = 0$. Using this decomposition, one introduces in \mathcal{H}^l the so-called canonical basis $\{|k, m \rangle\}$ where $k \geq |l|$ fixes the $SU(2)$ -representation, and $m = -k, \dots, k$ in integer steps. We have $m = 0$ if $k = 0$. The representation functions in the canonical basis are known; in particular, [19]

$$P^\rho(A) = \langle 0, 0 | T^{(0, \rho)}(A) | 0, 0 \rangle = \frac{2 \sin(\frac{1}{2} \eta \rho)}{\rho \sinh \eta} \quad (44)$$

for $A = u_1 d(\eta) u_2$, $u_i \in SU(2)$.

Lemma 6 *If $f(A) = f(u_1 A u_2)$ for all $u_1, u_2 \in SU(2)$, then*

$$\langle k, m | \tilde{F}(\chi) | k', m' \rangle = \delta_{l0} \delta_{k0} \delta_{k'0} \tilde{f}(\rho)$$

for some $\tilde{f}(\rho) \in \mathbf{C}$.

This is well known; cf. [19].

Using the lemma, the expansion simplifies,

$$f(A) = \int_{-\infty}^{\infty} \frac{\rho^2 d\rho}{8\pi} \tilde{f}(\rho) \langle 0, 0 | T^{(0, \rho)}(A) | 0, 0 \rangle.$$

The normalization of the Plancherel measure depends on the normalization of the Haar measure. We use the Haar measure (43) in the parametrization $A = u_1 d(\eta) u_2$.

In summary, the relevant representations $\Xi = (\rho)$ are labeled by one real parameter ρ ,

$$\int d\Xi \dots = \int_{-\infty}^{\infty} \frac{\rho^2 d\rho}{8\pi} \dots,$$

and the spherical function P^ρ is given by (44). Taking its complex conjugate, we see that the complex conjugate representation is $\bar{\Xi} = (-\rho)$.

Since the u_1, u_2 integrations are trivial, the inversion formula simplifies to the form used in Corollary 1.

Let us note that the inversion formula can also be obtained by elementary means, using the orthogonality of the sin-function,

$$\begin{aligned} & \int_{-\infty}^{\infty} d\rho \sin \frac{1}{2} \eta \rho \sin \frac{1}{2} \eta' \rho \\ &= 2\pi [\delta(\eta - \eta') - \delta(\eta + \eta')], \end{aligned}$$

and $\tilde{f}(-\rho) = \tilde{f}(\rho)$. In this way normalization factors can be fixed. The group theoretical interpretation is needed to make Lemma 4 available, though, in order to convert convolutions into products.

In applications we use the δ -function on $SL(2, \mathbf{C})$ with defining property

$$\int d\phi \delta(\phi) f(\phi) = f(\mathbf{1}).$$

Note that its normalization depends on the normalization of the Haar measure.

Harmonic analysis on $H = \mathbf{R}_+ SU(2)$

Elements $\Phi \in H = \mathbf{R}_+ SU(2)$ have the form $\Phi = ru$, with positive real r and $u \in SU(2)$. $SU(2)$ -bi-invariant functions $f(\Phi)$ depend only on r . Therefore harmonic analysis simplifies to a Mellin transform,

$$f(ru) = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} r^s \tilde{f}(s), \quad \tilde{f}(s) = \int_0^\infty dr r^{-s-1} f(ru).$$

Thus, the representations Ξ of H which appear in the expansion of $SU(2)$ -bi-invariant functions, are specified by pure imaginary numbers s , and the Haar measure and spherical function are

$$\int d\Xi \dots = \int \frac{ds}{2\pi i} \dots, \quad P^\Xi(ru) = r^{-s}.$$

The complex conjugate representation is $\bar{\Xi} = (-s)$.

Harmonic analysis on $H = GL(2, \mathbf{C})$

Elements of $GL(2, \mathbf{C})$ have the form $\Phi = \zeta A$, $0 \neq \zeta \in \mathbf{C}$, $A \in SL(2, \mathbf{C})$. Note that the pair $(-\zeta, -A)$ determines the same element as (ζ, A) , and that $SU(2)$ -bi-invariant functions f obey $f(-\zeta A) = f(\zeta A)$ because $(-\mathbf{1}) \in SU(2)$.

It follows that we have to combine harmonic analysis on $SL(2, \mathbf{C})$ with harmonic analysis on the multiplicative group $\mathbf{C} \setminus \{0\}$.

Consider functions g of $\zeta \in \mathbf{C} \setminus \{0\}$ with $g(-\zeta) = g(\zeta)$. Let $\zeta = e^{i\vartheta/2} r$, $r > 0$. Then

$$g(\zeta) = \sum_{l \in \mathbf{Z}} \int \frac{ds}{2\pi i} e^{il\vartheta} r^s \tilde{g}(l, s),$$

$$\tilde{g}(l, s) = \int_0^{2\pi} \frac{d\vartheta}{2\pi} dr e^{-il\vartheta} r^{-s-1} g(e^{i\vartheta/2} r).$$

Combining with harmonic analysis on $SL(2, \mathbf{C})$, we see that the representations Ξ of $GL(2, \mathbf{C})$ which appear in the expansion of $SU(2)$ -bi-invariant square integrable functions are labeled by $\Xi = (l, s, \rho)$, with $l \in \mathbf{Z}$, s imaginary, and ρ real, with Plancherel measure and spherical function

$$\int d\Xi \dots = \sum_{l \in \mathbf{Z}} \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \int_{-\infty}^{\infty} \frac{\rho^2 d\rho}{8\pi} \dots,$$

$$P^\Xi(e^{i\vartheta/2} r A) = e^{il\vartheta} r^s P^\rho(A),$$

where $P^\rho(A)$ is the spherical function on $SL(2, \mathbf{C})$.

The complex conjugate representation is labeled by $\bar{\Xi} = (-l, -s, -\rho)$.

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